Exercises 1: Review on Quantum Mechanics

1.- Spin-1/2 Heisenberg model for two sites: consider the Hamiltonian of two spins-1/2

$$H = \sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \sigma_1^z \sigma_2^z \ . \tag{1}$$

(a) Prove that $[H, \vec{S}] = 0$, where the total spin operator $\vec{S} = \vec{S}_1 + \vec{S}_2$ is defined such that

$$\vec{S}_1 \equiv \frac{1}{2}\left(\sigma_1^x, \sigma_1^y, \sigma_1^z\right) \qquad \vec{S}_2 \equiv \frac{1}{2}\left(\sigma_2^x, \sigma_2^y, \sigma_2^z\right) \tag{2}$$

(b) Compute the eigenbasis of operator $\sigma_1^z \sigma_2^z$, and write H in this basis as a matrix.

(c) Compute the eigenvalues E_i and eigenvectors $|E_i\rangle$ of H, and check that $H = \sum_i E_i |E_i\rangle\langle E_i|$.

(d) Given a quantum state $|\psi\rangle$ of several parties, the *reduced density matrix* ρ of a susbsystem is defined by taking the *partial trace* over all the other systems on the state projector $|\psi\rangle\langle\psi|$. For instance, for a quantum state $|\psi\rangle$ of two spins-1/2, the reduced density matrix of the first spin is given by

$$\rho_1 \equiv \mathrm{tr}_2 \left(|\psi\rangle\langle\psi|\right) = \sum_{i=1}^{2} \langle i_2|\psi\rangle\langle\psi|i_2\rangle \ , \tag{3}$$

where $\{|i_2\rangle\}$ is some basis for the Hilbert space of the second spin ($i_2 = 1, 2$). Also, the *von Neumann entropy or entanglement entropy* of a density matrix ρ is defined as

$$S(\rho) \equiv -\mathrm{tr}\left(\rho \log \rho\right) \ . \tag{4}$$

Compute ρ_1, ρ_2, $S(\rho_1)$ and $S(\rho_2)$ for the eigenstate $|E_0\rangle$ of H with the lowest energy E_0 (ground state), and prove that $S(\rho_1)$ and $S(\rho_2)$ are maximal over the space of 2×2 trace-one Hermitian matrices.

(e) Compute the time evolution operator $U(t) = \exp\left(-iHt\right)$.

(f) Compute the state $|\psi(t)\rangle = U(t)|\uparrow_1\rangle|\downarrow_2\rangle$, where $\{|\uparrow\rangle, |\downarrow\rangle\}$ are the eigenstates of σ^z with ± 1 eigenvalue (spin up/down in the z-basis).

(g) Compute $\langle\psi(t)|\sigma_1^z|\psi(t)\rangle$, $\langle\psi(t)|\sigma_1^x\sigma_2^z|\psi(t)\rangle$, $\langle\psi(t)|H|\psi(t)\rangle$, and check that $\langle\psi(t)|H|\psi(t)\rangle \geq E_0$ always, with E_0 the lowest energy eigenvalue of H. This is an example of the *variational principle* for the ground state energy of a Hamiltonian.

2.- Bell Basis and CHSH inequality: consider the four quantum states

$$\begin{aligned} |\psi^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow_1\downarrow_2\rangle \pm |\downarrow_1\uparrow_2\rangle\right) \\ |\phi^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow_1\uparrow_2\rangle \pm |\downarrow_1\downarrow_2\rangle\right) \end{aligned} \tag{5}$$

(a) Prove that they are an orthonormal basis of the Hilbert space of two spins-1/2. This is called the *Bell basis.*

(b) Compute ρ_1, ρ_2, $S(\rho_1)$ and $S(\rho_2)$ for these four states. Prove that $S(\rho_1)$ and $S(\rho_2)$ are always maximal over the space of 2×2 trace-one Hermitian matrices.

(c) Given the observables

$$\begin{aligned} Q &= \sigma_1^z \qquad S = \frac{-1}{\sqrt{2}}\left(\sigma_2^z + \sigma_2^x\right) \\ R &= \sigma_1^x \qquad T = \frac{1}{\sqrt{2}}\left(\sigma_2^z - \sigma_2^x\right) \end{aligned} \tag{6}$$

compute the quantity

$$|\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle| \tag{7}$$

where $\langle\ \cdot\ \rangle \equiv \langle\psi^-| \cdot |\psi^-\rangle$, and check that it is > 2.

(d) Compute the same combination of averages as above, but this time with $\langle\ \cdot\ \rangle \equiv \langle\uparrow_1\downarrow_2 | \cdot | \uparrow_1\downarrow_2\rangle$, and check that it is < 2. This is the *CHSH Bell inequality*: for classical (i.e. separable) bipartite probability distributions, the above combination of averages can *never* be larger than 2, whereas entangled quantum states can (and do) violate this. In fact, the states in the Bell basis maximally violate this inequality, and hence are also called *maximally entangled states.*