"Entanglement in many-body systems: concepts and algorithms"
Exercise Collection 2 – 29.04.2013

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A. Entanglement properties of a collection of singlets

Consider a system made of six spin-1/2 particles in the state $|\xi\rangle = |\psi^-\rangle_{12} \otimes |\psi^-\rangle_{34} \otimes |\psi^-\rangle_{56}$, where $|\psi^-\rangle_{ij} = (|\uparrow\rangle_i |\downarrow\rangle_j - |\downarrow\rangle_i |\uparrow\rangle_j)/\sqrt{2}$ is a maximally entangled Bell pair between two spins. Compute the Schmidt decomposition, the reduced density matrices and their entanglement (Von Neumann) entropies according to the following partitions:

a) $\{1\} \cup \{2...6\}$ b) $\{1,2\} \cup \{3...6\}$ c) $\{1,2,3\} \cup \{4,5,6\}$ d) $\{2,3\} \cup \{1,4...6\}$ e) $\{1,6\} \cup \{2...5\}$

Once performed the calculations, please focus on the following questions:

i) what relations can you notice among results? are some of them in particular ratios to others?

ii) if now many more spins are arranged on a 1D chain like $|\xi\rangle = |\psi^-\rangle_{12} \otimes \ldots \otimes |\psi^-\rangle_{2N-1,2N}$, what will be the entanglement entropy of the partitions $\{1, N\} \cup \{N+1 \ldots 2N\}$ and $\{1, 2N\} \cup \{2 \ldots 2N-1\}$? How does it scale with the number of spins N ?

iii) are you able to guess what will happen for a generalized situation in two or more dimensions, where the plane is covered by a collection of singlets? Any relation to concepts sketched in class? [*suggestion:* help yourself with a graphical representation of the Bell pairs and the partitions]

B. Spin-1 particles as composite objects

Consider two spin-1 particles, named A,B, and let their properties be ruled by the instance of a *bilinear-biquadratic* Hamiltonian with coefficients:

$$H = \left(\vec{S}_A \cdot \vec{S}_B\right) + \frac{1}{3}\left(\vec{S}_A \cdot \vec{S}_B\right)^2$$

i) what symmetries does H possess? i.e. which operators O are s.t. $[O, H] = 0$?

ii) what convenient form does H take once rewritten in terms of such symmetries? [*hint:* rewrite $\left(\vec{S}_A \cdot \vec{S}_B\right)$ in terms of the total spin $\vec{S}_{\text{tot}} = \vec{S}_A + \vec{S}_B$ and exploit its spectral form]

iii) what is the degeneracy of the ground manifold of H? i.e. how many degenerate ground states?

Now imagine that the spin-1 particles are composed by two spin-1/2 subparticles ($A = a \cup a'$, $B = b \cup b'$) whose paired state is "forced" to be a triplet ($\vec{S}_{A(B)} = \vec{S}_{a(b)} + \vec{S}_{a'(b')}$, $S_{A(B)} = 1$) instead of a singlet ($S_{A(B)} = 0$) [in spirit, this is very similar to isospin of nuclear states in terms of protons/neutrons, or to the composition of quarks to form hadronic matter] :

iv) rewrite the ground states of H in terms of the four spin-1/2 $a - a' - b - b'$;

v) compute their Schmidt decomposition with respect to the partition $\{a', b\} \cup \{a, b'\}$:
can you recognize a maximally entangled (Bell) pair in common to all of them?

vi) what is the total spin of the Bell pair appearing in all ground states? can you relate it (or better said, the possible states of the other two spin-1/2's) to the degeneration counting of iii)?

vii) if you pictorially represent a Bell pair with a solid segment between the spin-1/2's,
and the projectors $a \cup a' \to A$ ($b \cup b' \to B$) as circles encompassing the spin-1/2's,
does the resulting figure recall you some structure that was sketched in class? [*hint*: MPS?]