## Exercises 5: Toric Code and Topological Order

### 1.- Self-duality of the Toric Code on the square lattice

Consider the unitary *Hadamard transformation* defined as $H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$.

(a) How do $\sigma_x$ and $\sigma_z$ transform under $H$?

(b) How do the plaquette and star operators of the Toric Code transform under $H^{\otimes 4}$?

(c) How does the Hamiltonian of the Toric Code for $N$ sites transform under $H^{\otimes N}$?

(d) Prove that the quantum state

$$|+,+\rangle \equiv \prod_p \left( \frac{\mathbb{I} + B_p}{\sqrt{2}} \right) |+,+,...\rangle$$

is also a ground state of the Toric Code.

### 2.- Composite quasiparticle excitations

Consider the excitations of the Toric Code created by applying strings of $\sigma_y$ operators.

(a) What is the interpretation of these excitations in terms of charges and fluxes?

(b) What happens if we braid one of these $y$-quasiparticles around another $y$-quasiparticle? What is their statistics?

(c) What is the mutual statistics of these $y$-quasiparticles with charges? and with and fluxes?

### 3.- String tension and confinement

Consider the excitation spectrum of the Toric Code.

(a) Prove that all the excited states corresponding to two charges (or a pair charge/anticharge) have the same energy, no matter how far the charges are from each other. We say that the charges are *deconfined*.

(b) Next, consider adding a magnetic field in the $x$ and $z$ directions to the toric code Hamiltonian, i.e. our new Hamiltonian is

$$H = H_{TC} + h_x \sum_i \sigma_x^i + h_z \sum_i \sigma_z^i \quad .$$

This modification introduces a splitting in the energy states of the quasiparticle-pairs from (a). More specifically, the further away the quasparticles are, the higher is the energy cost. Therefore one says that there is a *string tension* between the quasiparticles, which is effectively equivalent to having a *confining potential* that tends to bond the quasiparticles together. This effect is more and more dramatic as the magnetic fields are increase in magnitude.

Consider the above scenario, and imagine that there are two quasiparticles (two charges or two fluxes) separated by a distance $L$. What should one expect if $L$ is too large? Does this remind you to some property of quarks in Quantum Chromodynamics?