Entanglement entropies of the Prime state

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Fundamental building blocks

NATURE

NUMBERS
In some sense quantum theory is a bending of physics towards number theory. However, deep facts of number theory play no role in questions of quantum mechanics....

In particular we do not know of any fundamental physical theories that are based on deep facts in number theory.

I would think that quantum mechanics will be completely reformulated and that number theory will play a key role in this formulation.

C. Vafa (2000)
Plan

- A primer of prime numbers
- Gas of primes, and Quantum Chaos and the zeta function
- Prime state: definition, construction, applications
- Entanglement properties of the Prime state
- Conclusions

Based on arXiv: 1302.6245 and work in progress
Prime counting function

\[ \pi(x) : \text{number of primes } p \text{ less than or equal to } x \]

\[ \pi(100) = 25 \]


Asymptotic behaviour: Gauss law

\[ \pi(x) \approx Li(x) \approx \frac{x}{\ln x} \]

\[ x \to \infty \]

Average behaviour or “mean field”
Prime Number Theorem (PNT)

$$\pi(x) \approx \text{Li}(x)$$

$$\text{Li}(x) = \int_2^x \frac{dt}{\log t} \approx \frac{x}{\log x} + \frac{x}{\log^2 x} + ...$$

Density of primes:

$$\frac{d\pi(x)}{dx} \approx \frac{1}{\ln x}$$

Largest known value

$$\pi(10^{24}) = 18\,435\,599\,767\,349\,200\,867\,886 \approx 1.8 \times 10^{22}$$

Platt (2012)

$$\text{Li}(10^{24}) - \pi(10^{24}) \approx 1.7 \times 10^{10}$$

The prime number function must oscillate around the Li(x) infinitely many times (Littlewood)

A first change of sign is expected for occur below the Skewes number

$$x < e^{727.9513468} ...$$
The fluctuations of $\pi(x)$ around $\text{Li}(x)$ are expected to be bounded by

$$|\text{Li}(x) - \pi(x)| < \frac{1}{8\pi} \sqrt{x} \log x$$

This statement is equivalent to the Riemann hypothesis (RH).
The zeta function and the Riemann hypothesis

Rosetta stone for Maths

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ Re } s > 1 \]
\[ \zeta(s) = \prod_{p=2,3,5,\ldots} \frac{1}{1-p^{-s}}, \text{ Re } s > 1 \]
\[ \zeta(s) = \frac{\pi^{s/2}}{2(s-1)\Gamma(1+s/2)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \]

n: integers
\[ \rho \rightarrow \text{ Riemann zeros} \]
p: primes

**Riemann hypothesis** (1859):

the complex zeros of the classical zeta function \( \zeta(s) \) all have real part equal to 1/2

\[ \zeta(s_n) = 0, s_n \in C \rightarrow s_n = \frac{1}{2} + i E_n, \quad E_n \in \mathbb{R}, n \in \mathbb{Z} \]

In fact:

\[ |Li(x) - \pi(x)| \leq \frac{1}{8\pi} \sqrt{x} \log x \]
A gas of primes (Julia, Spector, 1990)

Single particle levels

\[ H \left| p \right> = E_0 \log p \left| p \right>, \quad p = 2, 3, 5, \ldots \]

Many particle state of bosons

\[ \left| n \right> = \left| k_2, k_3, \ldots, k_p, \ldots \right> \]

\[ n = 2^{k_2} 3^{k_3} \ldots p^{k_p} \ldots \]

Total energy

\[ E(n) = E_0 \sum_p k_p \log p = E_0 \log n \]

Partition function

\[ Z = \sum_n \exp \left( -\frac{E(n)}{k_B T} \right) = \sum_n \exp \left( -\frac{E_0 \log n}{k_B T} \right) = \sum_n \frac{1}{n^s} = \zeta(s) \]

\[ s = \frac{E_0}{k_B T} \]

Z diverges at s=1 → Hagedorn transition
The Riemann zeros look like the spectrum of random physical systems

Bohigas, Gianonni (1984)
Quantum chaos, primes and Riemann zeros

Berry Conjecture (1986): there exists a chaotic Hamiltonian \( H \)

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Proof of the Riemann hypothesis
In previous models the primes are classical objects:

- Energies of Stat Mech (gas model)
- Periods of orbits (quantum chaos)

Idea: make the primes quantum objects
Quantum Computation and prime numbers
(JLT, GS, 2013)

Classical computer

$n$ bits

\[ x = x_0 2^0 + x_1 2^1 + \ldots + x_{n-1} 2^{n-1}, \quad x_i = 0,1, \quad x = 0,1, \ldots 2^n - 1 \]

Quantum computer

$n$ qubits

\[ |x\rangle = |x_{n-1}, \ldots, x_0\rangle = |x_{n-1}\rangle \otimes \ldots \otimes |x_0\rangle \]
The Prime State

Primes $\rightarrow$ State

$|P(n)\rangle = \frac{1}{\sqrt{\pi(2^n)}} \sum_{p < 2^n \in \text{Primes}} |p\rangle$

$\pi(2^n)$ is the prime counting function
Quantum Mechanics allows for the superposition of primes implemented as states of a computational basis

\[ |P(n)\rangle = \frac{1}{\sqrt{\pi (2^n)}} \sum_{p < 2^n \in \text{Primes}} |p\rangle \]

Ex. \( n=3 \)

\[ |P(3)\rangle = \frac{1}{\sqrt{4}} (|2\rangle + |3\rangle + |5\rangle + |7\rangle) \]
Could the Prime state be constructed?

Does it encode properties of prime numbers?

Could it provide the means to explore Arithmetics?

What are its entanglement properties?
Construction of the Prime state

\[ U_{\text{primality}} \sum_x |x\rangle |0\rangle = |P(n)\rangle |0\rangle + \sum_{c \in \text{composite}} |c\rangle |1\rangle \]

\[ \text{Prob}(|P(n)\rangle) = \frac{\pi (2^n)}{2^n} \approx \frac{1}{n \log 2} \]

Efficient construction

PNT
Grover construction of the Prime state

\[ |\psi_0\rangle = \sum_{x < 2^n} |x\rangle = \frac{1}{\pi(2^n)} \left( \sum_{p \in \text{primes}} |p\rangle + \sum_{c \in \text{composites}} |c\rangle \right) \]

# calls to the oracle

\[ R(n) \leq \frac{\pi}{4} \sqrt{\frac{N}{M}} \leq \frac{\pi}{4} \sqrt{n \log 2} \]

\[ |\psi_f\rangle \approx |P(n)\rangle \]
# calls to Grover

Overlap between Grover state and the Prime state

We need to construct an oracle!
Construction of a Quantum Primality oracle

An efficient Quantum Oracle can be constructed using classical primality tests

**Miller-Rabin primality test**

- Find $s$ and $d$ (odd) such that
  \[ x \rightarrow x - 1 = 2^s d \]
- Choose witness $a$ \( 1 \leq a \leq x \)

- If \( a^d \neq 1 \mod x \) then \( x \) is composite with certainty
  \[ a^{2^r d} \neq -1 \mod x \quad 0 \leq r \leq s - 1 \]

- If the test fails, \( x \) may be prime or composite.
- Latter case: \( a \) is a strong liar to \( x \)
- Eliminate strong liars checking less than \( (\log x)^2 \) witnesses
Finding $d$ and $s$

$$x = 49 \rightarrow x - 1 = 48 = 2^4 \times 3 \rightarrow s = 4, d = 3$$

$$|48\rangle = |1,1,0,0,0,0,0\rangle$$

$d$ $s$
Structure of the quantum primality oracle
Tests are condition to the actual value of $x$. 

Tests:
- $s=1$: $|d>=|x_3x_2x_1>$
- $s=2$: $|d>=|x_3x_2>$
- $s=3$: $|d>=|x_3>$

Modular Exponentiation qubits

Test carriers
Quantum Counting of Prime numbers

quantum primality oracle + quantum counting algorithm

Brassard, Hoyer, Tapp (1998)

Counts the number of solutions to the oracle
We want to count $M$ solutions out of $N$ possible states

We know an estimate $\tilde{M}$

$$|\tilde{M} - M| < \frac{2\pi}{c} M^{\frac{1}{2}} + \frac{\pi^2}{c^2}$$

Bounded error in quantum counting

Bounded error in the quantum counting of primes

$$\left| \pi_{QM}(x) - \pi(x) \right| \leq \frac{2\pi}{c} \frac{x^{1/2}}{\log^{1/2} x}$$

We use the PNT
\[ \left| \pi_{QM}(x) - \pi(x) \right| \leq \frac{2\pi}{c} \frac{x^{1/2}}{\log^{1/2} x} \]

Riemann Hypothesis
\[ \left| Li(x) - \pi(x) \right| < \frac{1}{8\pi} \sqrt{x} \log x \]

Error of quantum counting < fluctuations under the RH

A quantum computer could falsify the RH, but not prove it !!

Historical remark:
In 1950 Turing was the first person to use an electronic computer of the Manchester University to find the first 1104 Riemann zeros, then the machine broke down.

Non believers in the RH should build a quantum computer.
Turing gear-driven mechanical calculator (1939)
A Quantum Computer could calculate the size of fluctuations more efficiently than a classical computer.


$$T \sim x^{\frac{1}{2}} \quad S \sim x^{\frac{1}{4}}$$

A Quantum Computer could calculate the size of fluctuations more efficiently than a classical computer.

$$T \sim x^{\frac{1}{2}} \quad S \sim \log x$$
Entanglement of a single qubit:

\[
|P(n)\rangle = \frac{1}{\sqrt{\pi (2^n)}} \sum_{i_{n-1}, \ldots, i_1, i_0 = 0,1} p_{i_{n-1} \ldots i_1i_0} |i_{n-1}, \ldots, i_1, i_0\rangle
\]

\[
p_{i_{n-1} \ldots i_1i_0} = \begin{cases} 
1 & p=i_{n-1}2^{n-1}+\ldots+i_0 = \text{prime} \\
0 & \text{otherwise}
\end{cases}
\]

Density matrix qubit $i=1$

\[
\rho_{ab}^{(1)} = \frac{1}{\pi (2^n)} \sum_{i_{n-1}, \ldots, i_2, i_0 = 0,1} p_{i_{n-1} \ldots i_2a_i_0} p_{i_{n-1} \ldots i_2b_i_0}
\]

\[
\rho_{00}^{(1)} = \frac{\pi_{4,1} (2^n)}{\pi (2^n)}
\]

\[
\rho_{11}^{(1)} = \frac{1+\pi_{4,3} (2^n)}{\pi (2^n)}
\]

\[
\rho_{01}^{(1)} = \frac{\pi_{2}^{(1)} (2^n)}{\pi (2^n)}
\]

Odd primes

\[
\pi_{4,1} : 5, 13, 17, \ldots 4n + 1
\]

\[
\pi_{4,3} : 3, 7, 11, \ldots 4n + 3
\]
Dirichlet theorem:
There infinite number of primes of the form $1 + 4n$ and $3 + 4n$

PNT for arithmetic series

\[
\lim_{x \to \infty} \frac{\pi_{4,1}(x)}{Li(x)} = \lim_{x \to \infty} \frac{\pi_{4,3}(x)}{Li(x)} = \frac{1}{\phi(4)} = \frac{1}{2} \quad \rightarrow \quad S\left(\rho^{(i)}\right) \sim \log 2
\]

\[
\pi_{1,31}(N)/Li(N)
\]

\[
\begin{align*}
3 \mod 4 \\
1 \mod 4
\end{align*}
\]
Chebyshev bias:

For low values of $x$ there are more primes $3 \mod 4$ than $1 \mod 4$

$$\Delta(x) = \pi_{4,3}(x) - \pi_{4,1}(x)$$

Chebyshev bias gives the magnetization of qubit $i=1$

$$\langle \sigma_z^{(1)} \rangle = \frac{-\Delta(2^n) - 1}{\pi(2^n)}$$
Counting twin primes: $p, p+2$

Counting $p \leq x$, $p+2$ prime

\[ \pi_2(x) \approx 2C_2 \frac{x}{(\log x)^2} \]

Twin prime constant

\[ C_2 = \prod_{p \geq 3} \frac{p(p-2)}{(p-1)^2} = 0.6601.. \]

Counting $p \leq x$, $p+k$ prime:

\[ \pi_2(k, x) \approx C(k) \frac{x}{(\log x)^2} \]

$k=2,4,6$
Bias in the twin primes

Twin primes

$\pi_2^{(1)} : (5, 7), \ldots \quad (1 \mod 4, 3 \mod 4)$

$\pi_2^{(3)} : (11, 13), \ldots \quad (3 \mod 4, 1 \mod 4)$

Can be measured by off diagonal correlations

$$
\langle \sigma_x^{(1)} \rangle = \frac{2 \pi_2^{(1)} (2^n)}{\pi (2^n)}, \quad \langle \sigma_x^{(1)} \sigma_x^{(2)} + \sigma_y^{(1)} \sigma_y^{(2)} \rangle = \frac{4 \pi_2^{(3)} (2^n)}{\pi (2^n)}
$$

Twinship $\rightarrow$ off diagonal entries of density matrix

Sub-series of primes, twin primes, etc. are amenable to measurements
Entanglement entropy of the Prime state

\[ S \sim 0.8858 n + \text{const} \]
Reduced Density matrix

\[ \rho(i, j) \approx \delta_{i,j} + \frac{1}{n \log 2} C(2|i - j|) \]

\[ \pi_2(k, x) \approx C(k) \frac{x}{(\log x)^2} \]

Suggest that

\[ S \sim 0.8858n + \text{const} \]

Some deep property of prime numbers
Scaling of entanglement entropy

\[ S \sim n - \text{const} \quad \text{Random states} \]

\[ S \sim 0.8858 \, n + \text{const} \quad \text{Prime state} \]

\[ S \sim n^{\frac{d-1}{d}} + \text{const} \quad \text{Area law in } d\text{-dimensions} \]

\[ S \sim \frac{c}{3} \log n + \text{const} \quad \text{Critical scaling in } d=1 \text{ at quantum phase transitions} \]

\[ S \sim \log (\xi) = \text{const} \quad \text{Finitely correlated states away from criticality} \]
Original partition carries more entanglement (!?)

Entanglement entropy for different partitions

\[ \mu = 5.72307; \quad \sigma = 0.0184293; \quad \frac{S_{\text{max}} - \mu}{\sigma} = 3.47743 \]
Construction of the Twin Prime state

\[ U_{+2} | P(n) \rangle | 0 \rangle = \sum_{p \in \text{primes}} | p+2 \rangle | 0 \rangle \]

\[ U_{\text{primality}} U_{+2} | P(n) \rangle | 0 \rangle = \sum_{p, p+2 \in \text{primes}} | p+2 \rangle | 0 \rangle + \sum_{p+2 \notin \text{primes}} | p+2 \rangle | 1 \rangle \]

\[ \text{Pr}(|\text{twin primes}\rangle) = \frac{\pi_2(2^n)}{\pi(2^n)} \approx \frac{2C_2}{n \log 2} \]
Conclusions

- The Prime state provides a link between Number Theory and Quantum Mechanics

- Quantum Computers could be used as Quantum Simulators of Arithmetics

- Arithmetic properties could be measured more efficiently than with classical algorithms, e.g. to falsify the Riemann hypothesis

- Entanglement in the Prime state captures fundamental properties as pair correlations between the primes

- Possible connections with Random Matrix Theory, Quantum Chaos and the Riemann zeros
Thank you