Exercises 2: Basic Quantum Engineering

1.- Quantum cloning of orthogonal states: We have seen in class that general quantum states can not be cloned. However, it is possible to clone *states that are orthogonal.*

(a) Build a two-qubit unitary transformation U_a that clones the states $|0_1\rangle$ and $|1_1\rangle$ of the first (register) qubit into the second (target) one. What are its matrix elements?

(b) What happens if you try to clone the states $|+_1\rangle$ and $|-_1\rangle$ with this operator? What would be the matrix elements of a unitary operator U_b cloning these two states?

(c) The *operator fidelity* between unitaries U and W is defined as $F(U,W) \equiv |\mathrm{tr}(UW^\dagger)|$. This quantity satisfies $0 \leq F(U,W) \leq 1$, and measures the similitude between U and W (in a way, it can be understood as a "scalar product" between unitary operators). Compute $F(U_a, U_b)$, and interpret the result.

2.- Superdense coding with qudits: Let us consider d-level systems defined by a Hilbert space basis $\{|0\rangle, |1\rangle, |2\rangle, ..., |d\rangle\}$. These are called *qudits*, and are the generalization to higher-dimensional systems of the usual qubits.

(a) Consider the d^2 quantum states for two qudits

$$|\phi_{lp}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{2\pi i l j/d} |j \oplus p\rangle |j\rangle \ ,$$

where $\oplus$ is the "sum modulo d", i.e. $j \oplus p =$ the remainder of $(j+p)/d$ (equivalent to wrapping the integers around a circle with d sites). Prove that these states form an orthogonal basis of the Hilbert space of two qudits, that is, $\langle \phi_{lp} | \phi_{l'p'} \rangle = \delta_{ll'} \delta_{pp'}$. This is the *Bell basis for qudits.*

(b) Alice and Bob share the entangled state $|\phi_{00}\rangle$ (Alice has one qudit, and Bob has the other). At some point, Alice wants to send 2 classical "dits" (l, p) of information to Bob using a superdense coding protocol. Which unitary operators U_{lp} should Alice apply to her qudit, before sending it to Bob? (Hint: find an operator such that $U_{lp} |\phi_{00}\rangle = |\phi_{lp}\rangle$).

3.- Quantum repeaters: Remember the Bell Basis of two qubits from Exercise 1.2, given by four maximally entangled states $\{|\phi^\pm\rangle, |\psi^\pm\rangle\}$.

(a) Consider a quantum state of four qubits given by $|\Psi\rangle = |\phi^+_{12}\rangle|\phi^+_{34}\rangle$ (the subscript refers to the qubit). We do a *Bell measurement* in the qubits 2 and 3, that is to say, we measure the value of the quantum state for these two qubits in the Bell basis. As a result, the wave function for qubits 2 and 3 collapses in one of the four states $\{|\phi^\pm_{23}\rangle, |\psi^\pm_{23}\rangle\}$. Imagine that the outcome of the measurement is the state $|\phi^+_{23}\rangle$. After the collapse of the wavefunction, what is the overall quantum state of the four qubits? Is there anything special about the remaining state for qubits 1 and 4? What happens if the outcome of the measurement is any of the other states in the Bell basis? This procedure is called *entanglement swapping*.

(b) Now we have a quantum state of $2n$ qubits given by $|\Psi\rangle = |\phi^+_{12}\rangle|\phi^+_{34}\rangle \cdots |\phi^+_{(2n-1)2n}\rangle$. If we do Bell measurements on the qubit pairs $(2,3), (4,5), \ldots, (2n-2, 2n-1)$, should we expect something special about the remaining state for qubits 1 and $2n$? Try to give an intuitive explanation in terms of pictures, representing qubits by circles and Bell states by lines between circles. The places where we do Bell measurements are called *quantum repeaters*, and allow to extend entanglement to arbitrarily long distances in physical space.