Exercises 3: Quantum Circuits

1.- Single-qubit gates:

(a) Let x be a real number and A a matrix such that $A^2 = I$. Show that

$$e^{iAx} = \cos(x) I + i \sin(x) A \ .$$

Use this to verify the expressions for $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$ given in class.

(b) Show that, up to a global phase, the $\pi/8$ gate satisfies $T = R_z(\pi/4)$.

(c) Show that $XYX = -Y$, and use this to prove that $XR_y(\theta)X = R_y(-\theta)$.

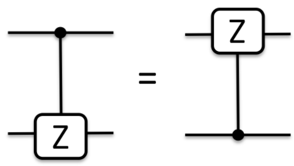
(d) Show that $HXH = Z$, $HYH = -Y$, and $HZH = X$.

(e) Use the previous result to show that $HTH = R_x(\pi/4)$, up to a global phase.

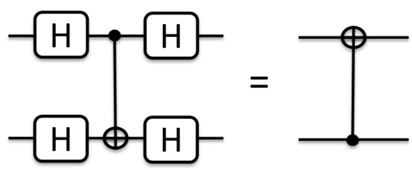
2.- Controlled gates:

(a) Construct a quantum circuit to obtain a CNOT gate from one $C(Z)$ gate and two Hadamard gates, specifying the control and target qubits.

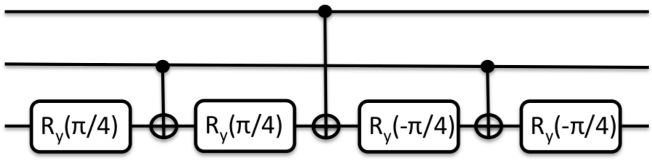
(b) Show that



(c) Show that



(d) Show that the circuit:

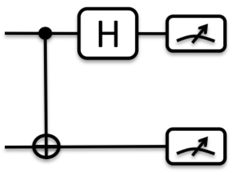


differs from a Toffoli gate only by relative phases. That is, the circuit takes $|c_1, c_2, t\rangle$ to $e^{i\theta(c_1,c_2,t)}|c_1, c_2, t \oplus c_1 \cdot c_2\rangle$, where $e^{i\theta(c_1,c_2,t)}$ is some relative phase factor. Such gates can sometimes be useful in experimental implementations.

(e) Let subscripts denote which qubit an operator acts on, and let C be a CNOT with qubit 1 the control qubit and qubit 2 the target qubit. Prove the following identities:

$$\begin{aligned} CX_1C &= X_1X_2 \\ CY_1C &= Y_1X_2 \\ CZ_1C &= Z_1 \\ CX_2C &= X_2 \\ CY_2C &= Z_1Y_2 \\ CZ_2C &= Z_1Z_2 \\ R_{z,1}(\theta)C &= CR_{z,1}(\theta) \\ R_{x,2}(\theta)C &= CR_{x,2}(\theta) \end{aligned}$$

3.- Measurements in the Bell basis: The measurement model we have specified for the quantum circuit model is that measurements are performed only in the computational basis $\{|0\rangle, |1\rangle\}$. However, often we want to perform a measurement in some other basis, defined by a complete set of orthogonal states. To perform this measurement, simply unitarily transform from the basis we wish to perform the measurement in to the computational basis, and then measure. For example, show that the circuit

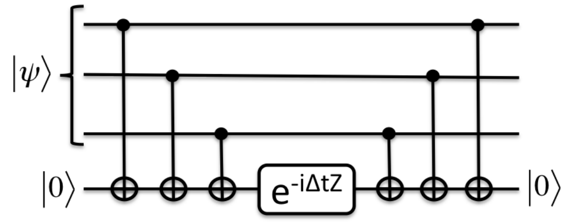


performs a measurement in the basis of the Bell states $\{|\phi^{\pm}\rangle, |\psi^{\pm}\rangle\}$ from Exercise 1.1.

4.- Quantum simulation: Suppose we have the Hamiltonian of 3 qubits $H = X_1 \otimes Y_2 \otimes Z_3$.

(a) Show that $H = (U_1 \otimes U_2 \otimes U_3)\widetilde{H}(U_1^\dagger \otimes U_2^\dagger \otimes U_3^\dagger)$, with U_1, U_2, U_3 some unitary operators acting respectively on each one of the qubits, and $\widetilde{H} = Z_1 \otimes Z_2 \otimes Z_3$. What is the value of U_1, U_2 and U_3?

(b) Show that the quantum circuit



exactly produces the state $e^{-i\Delta t \widetilde{H}}|\psi\rangle$ for an input state $|\psi\rangle$ of the three upper qubits. Therefore, it *simulates* a time evolution driven by Hamiltonian $\widetilde{H}$ for a time Δt on these three qubits.

(c) Show that for an unitary operator U and a matrix A,

$$e^{UAU^\dagger} = U e^A U^\dagger \ . \tag{1}$$

Use this property and the results from the previous two sections to build a quantum circuit simulating a time evolution driven by Hamiltonian H for a time Δt on three qubits.

(d) Prove the Trotter-Suzuki formula

$$e^{i(A+B)\Delta t} = e^{iA\Delta t} e^{iB\Delta t} + O(\Delta t^2) \ ,$$

and use it to construct a quantum circuit simulating the time evolution for a time Δt of a 4-qubit system with Hamiltonian $H = X_1 \otimes Y_2 \otimes Z_3 \otimes I_4 + I_1 \otimes X_2 \otimes Y_3 \otimes Z_4$, up to an error $O(\Delta t^2)$. How would you simulate an evolution for a time $T = n\Delta t$, with small error?

(e) Can you think of a strategy to simulate an arbitrarily long time evolution, driven by an arbitrary Hamiltonian, and acting on arbitrarily many qubits?

5.- Heisenberg interaction: The Heisenberg Hamiltonian between two qubits is

$$H = \lambda/4(X_1X_2 + Y_1Y_2 + Z_1Z_2)\ . \qquad (2)$$

This is a very common Hamiltonian in the physics of magnetism, and we would like to use it to make two-qubit gates for quantum computation.

(a) Show that one can produce a $C(Z)$ gate by the following sequence, involving two Heisenberg interactions:

$$C(Z) = e^{i\pi Z_1/4}e^{-i\pi Z_2/4}U_{sw}^{1/2}e^{i\pi Z_1/2}U_{sw}^{1/2}\ , \qquad (3)$$

where $U_{sw}^{1/2}$ is the "square root swap" operator, defined as $U_{sw}^{1/2} = e^{itH}$, with $\lambda t = \pi/2$.

(b) Show how to convert a CNOT gate into a $C(Z)$ gate by using two Hadamard gates on one of the qubits.