Exercises 5: Quantum error correction, and other quantum computer models (theory chapters 8 and 9)

1.- Fault-tolerant gates: find a fault-tolerant version of the CNOT gate for the 3-qubit flip code, the 3-qubit phase flip code, and the 9-qubit Shor code (hint: be careful about how errors propagate between the qubits!)

2.- Classical self-correcting magnetic memories: independently of the corrections of errors along a computation, sometimes it is useful to store information (either classical or quantum) in a memory device. For such a storage to be reliable, it must be robust against errors. In particular, one wishes to (i) store the information, (ii) forget about it for a long time, and then (iii) recover the information when needed at a later stage. Therefore, the memory must be robust in the sense that, when doing *nothing*, the information must not be corrupted in time to the point of not being recoverable.

Imagine now that you are given a classical ferromagnetic material described by the Hamiltonian

$$H = \sum_{\langle i,j \rangle} s_i s_j \ , \tag{1}$$

with $s_i = \pm 1$ being classical spins. This is the so-called *Ising model* of ferromagnetism, and is a valid model for many magnetic materials.

(a) According to statistical mechanics, at an inverse temperature β the probability of a given configuration S of the spins is given by $p(S) = e^{-\beta E(S)}/Z$, with $E(S)$ its energy and Z the partition function. Moreover, the probability of jumping from one configuration S to another S' with higher energy is given $p(S \to S') = e^{-\beta(E(S')-E(S))}$. Assume now that all the spins are initially in the $+1$ state. For a 1d ferromagnet with nearest-neighbor interactions, consider the alternative configuration where a large domain of contiguous spins has been flipped to the -1 state. How much is the probability of jumping from the initial configuration to the "flipped" one? How does this scale with the number of spins flipped?

(b) Repeat the above for a $2d$ ferromagnet with nearest-neighbor interactions on a square lattice. Is there any fundamental difference with respect to 1d? What about 3d?

(c) How would you design a *robust classical memory for one bit* using such a system? This is an example of a *self-correcting memory*, where errors are corrected automatically by the interactions in the system. *Self-correcting quantum memories* also exist, e.g. the 4d Toric Code model, but are based on very different principles.

3.- Adiabatic quantum search: Consider an N-dimensional Hilbert space, where we define the Hamiltonians $H_0 = \mathbb{I} - |\psi\rangle\langle\psi|$ and $H_m = \mathbb{I} - |m\rangle\langle m|$, with $|\psi\rangle = (1/\sqrt{N})\sum_{i=1}^{N}|i\rangle$ and $|m\rangle$ some 'marked" vector. H_m is an oracle Hamiltonian, which is the Hamiltonian version of the oracle operator in Grover's quantum search algorithm. We now construct the Hamiltonian

$$H(s) = (1-s)H_0 + sH_m \ , \tag{2}$$

where $s = s(t)$ is a time-dependent function obeying the constraints of adiabatic quantum computation, i.e. $s(0) = 0$ and $s(T) = 1$ for a given running time T.

(a) Prove that the two lowest eigenvalues of $H(s)$ are given by

$$E_{\pm}(s) = \frac{1}{2}\left(1 \pm \sqrt{(1-2s)^2 + \frac{4}{N}s(1-s)}\right) \ , \tag{3}$$

and therefore the gap is given by $\Delta = \sqrt{1 - 4\frac{N-1}{N}s(1-s)}$. According to the global adiabatic condition

$$\frac{\left|\langle \frac{dH}{dt} \rangle_{1,0}\right|}{\Delta^2_{min}} \leq \epsilon \ , \tag{4}$$

prove that if $s(t) = t/T$, then the total running time of the algorithm is given by $T \geq N/\epsilon$, with ϵ the final error. Thus, no speedup is present as compared to classical methods.

(b) We now wish to apply the above adiabatic condition not globally, but locally to each infinitesimal time step, i.e: the smaller the instantaneous gap, the slower the instantaneous evolution. Prove that the above adiabatic condition can be reformulated as

$$\left|\frac{ds}{dt}\right| \leq \epsilon \frac{\Delta^2(t)}{\left|\langle \frac{dH}{ds} \rangle_{1,0}\right|} \tag{5}$$

for all times t. Show that this condition is in fact satisfied by

$$\frac{ds}{dt} = \epsilon\Delta^2(t) = \epsilon\left(1 - 4\frac{N-1}{N}s(1-s)\right) \ . \tag{6}$$

Integrate the above equation, and prove that

$$t = \frac{1}{2\epsilon}\frac{N}{\sqrt{N-1}}\left(\arctan\left(\sqrt{N-1}(2s-1)\right) + \arctan\sqrt{N-1}\right) \ . \tag{7}$$

Finally, show that for $s = 1$ and $N \gg 1$, one obtains a total running time $T = \frac{\pi}{2\epsilon}\sqrt{N}$, thus achieving a square-root speed-up with respect to classical means.

4.- Cluster State and Cluster Hamiltonian:

We have seen that the Cluster State $|\psi_c\rangle$ is defined uniquely on a lattice by the set of equations

$$T_i|\psi_c\rangle = (+1)|\psi_c\rangle \ , \tag{8}$$

where T_i are the so-called stabilizer operators

$$T_i = X_i \bigotimes_{n(i)} Z_{n(i)} \ , \tag{9}$$

with $n(i)$ the nearest-neighbors of lattice site i.

(a) Prove that the stabilizer operators are always mutually conmuting, i.e.

$$[T_i, T_j] = 0 \quad \forall i, j \tag{10}$$

(b) Use the above property to prove that the Cluster State is the ground state of the many-body Cluster Hamiltonian

$$H = -\sum_i T_i \tag{11}$$

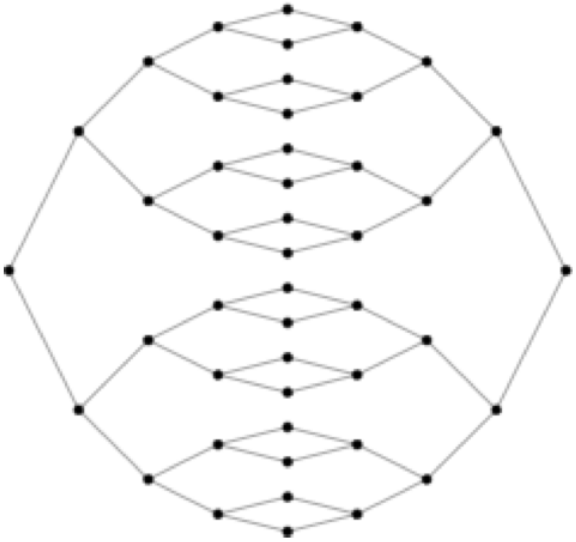
(c) Prove that, up to normalization, the cluster state can be written also as

$$|\psi_c\rangle = \prod_i P_i|\phi\rangle, \tag{12}$$

with $P_i \equiv (\mathbb{I} + T_i)/2$ a projector, and $|\phi\rangle$ some reference state of the qubits. Use this property to compute the Cluster State for 3 sites on a triangle in the computational basis for each qubit.

5.- Quantum walk on glued trees:

Consider the following graph, consisting of two glued binary trees:



(a) Let us define a classical random walk on the graph. For a given node, the walker has equal probability of jumping to any of the nearest-neighboring nodes. If the walker is in a given node correspoding to a given column, what is the probability of moving to the left? And to the right? For very long times, what is the most probable behavior of the walker?

(b) Now we define a quantum walk on the graph, driven by the Hamiltonian

$$\langle a|H|b\rangle = M_{ab} \ , \tag{13}$$

with M_{ab} the adjacency matrix of the graph ($M_{ab} = 1$ if a and b are nearest neighbors, and zero otherwise). Consider the basis of n column states

$$|\text{col } j\rangle = \frac{1}{\sqrt{N_j}} \sum_{a \in \text{column}(j)} |a\rangle \qquad j = 1, 2, \ldots, n. \tag{14}$$

Compute the normalization factors N_j, the non-zero matrix elements $\langle \text{col } j|H|\text{col } j'\rangle$, and interpret the result in terms of a quantum walk *on a line*. If the quantum walker starts from the left-most node, what will be (intuitively) its behavior after a time evolution for a time $T \sim O(n)$? For large n, this is a simple example of an *exponential speed-up* in transversing a graph by means of a walk.