Quantum Hall Effect and Topological Order, Exercises 1

1.- Classical Hall Effect:

(a) The classical equation of motion for a particle of mass m and charge $-e$ in a magnetic field is

$$m\frac{d\vec{v}}{dt} = -e\vec{v} \times \vec{B}. \tag{1}$$

For a magnetic field in the z-direction, so that $\vec{B} = (0, 0, B)$, prove that the general solution is

$$x(t) = X - R\sin(\omega_B t + \phi) \quad \text{and} \quad y(t) = Y + R\cos(\omega_B t + \phi), \tag{2}$$

with $\omega_B = eB/m$ the *cyclotron frequency*. What is the physical meaning of X and Y?

(b) If on top opf $\vec{B}$ we add an electric field $\vec{E}$ and a friction term τ, the resulting equation of motion is

$$m\frac{d\vec{v}}{dt} = -e\vec{E} - e\vec{v} \times \vec{B} - \frac{m\vec{v}}{\tau}. \tag{3}$$

The above is called the *Drude model*, and it is the simplest (classical) model of charge transport. Prove that in the stationary regime, i.e., when $d\vec{v}/dt = 0$, one can write the above equation as

$$\vec{J} = \sigma\vec{E}, \tag{4}$$

with $\vec{J} = -ne\vec{v}$ the current density (n is the electron density), and σ the *conductivity tensor* (a 2×2 matrix). Prove also that

$$\sigma = \frac{\sigma_{DC}}{1 + \omega_B^2\tau^2}\begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix}, \tag{5}$$

with $\sigma_{DC} = ne^2\tau/m$ the DC conductivity in the absence of a magnetic field. What is the meaning of the off-diagonal components?

(c) Define the resistivity tensor as $\rho = \sigma^{-1}$. What is the behaviour of the diagonal and off-diagonal components as a function of the intensity of the magnetic field? How does this compare to the experimental results mentioned in class?

2.- Poisson brackets and commutators:

(a) The classical poisson bracket for two functions $f(q,p)$ anf $g(q,p)$ of canonical coordinates q and p is defined as

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \tag{6}$$

For a classical particle in a magnetic field, we define the vector function $\vec{\pi} = \vec{p} + e\vec{A}$, where $\vec{p}$ is the canonical momentum and $\vec{A}$ the the vector potential. Prove that

$$\{\pi_i, \pi_j\} = -e\epsilon_{ijk}B_k. \tag{7}$$

(b) In the quantum case, functions are promoted to operators and Poisson brackets are promoted to commutators. Prove that for a charged quantum particle on a magnetic field, this implies that the quantum operators

$$a = \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y) \tag{8}$$

obey the commutator algebra of a quantum harmonic oscillator, i.e., $[a, a^\dagger] = 1$.

3.- Landau levels on an electric field: As explained in class, the Hamiltonian of an electron with a magnetic field in the z-direction and an electric field in the x-direction can be written in the Landau gauge as

$$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right) - eEx. \tag{9}$$

Prove that this is simply the Hamiltonian of a displaced harmonic oscillator, with eigenfunctions

$$\psi(x,y) = \psi_{n,k}(x - mE/eB^2, y) \tag{10}$$

and eigenenergies

$$E_{n,k} = \hbar\omega_B\left(n + \frac{1}{2}\right) + eE\left(kl_B^2 - \frac{eE}{m\omega_B^2}\right) + \frac{m}{2}\frac{E^2}{B^2}. \tag{11}$$