Quantum Hall Effect and Topological Order, Exercises 2

1.- A spin in a magnetic field: Consider a spin-1/2 particle in a magnetic field $\vec{B}$ with Hamiltonian

$$H = -\vec{B} \cdot \vec{\sigma} + B, \tag{1}$$

with $\vec{\sigma}$ a vector of Pauli matrices. The constant term in the Hamiltonian ensures that the ground state energy is always 0.

(a) Prove that the ground state has energy 0, and that the excited state has energy $2B$. Let's call $|\downarrow\rangle$ the ground state, and $|\uparrow\rangle$ the excited state.

(b) Parametrise the magnetic field in spherical coordinates in terms of 2 angles. Write the resulting 2×2 matrix for H, as well as its eigenstates.

(c) Calling $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ the two relevant angles in the parametrization above, prove that the Berry curvature in polar coordinates is given by

$$\mathcal{F}_{\theta\phi} = \frac{\partial \mathcal{A}_\phi}{\partial \theta} - \frac{\partial \mathcal{A}_\theta}{\partial \phi} = -\sin\theta. \tag{2}$$

(d) What is the Berry phase for the path $\theta = \pi/2, \phi \in [0, 2\pi]$?

(e) Prove that the integral of the Berry curvature over any sphere S^2 surrounding the origin is

$$\int_{S^2} \mathcal{F}_{ij} dS^{ij} = -2\pi. \tag{3}$$

What is the corresponding Chern number?

2.- Spectral flow: A vector potential $\vec{A}$ can affect the physics of a quantum particle, even if it moves outside of the region where $\vec{B} \neq 0$. To see how this happens, imagine a cylinder (solenoid) carrying a magnetic field $\vec{B}$ along its tube. If the section of the cylinder is A, then it carries a magnetic flux $\Phi = BA$. Outside the solenoid, the magnetic field is zero, but the vector potential is not.

(a) Stokes' theorem implies that the line integral of the vector potential outside the solenoid is given by

$$\oint \vec{A} \cdot d\vec{r} = \int \vec{B} \cdot d\vec{S} = \Phi. \tag{4}$$

Prove that a solution to this equation in cylindrical polar coordinates is given by

$$A_\phi = \frac{\Phi}{2\pi r}. \tag{5}$$

(b) Consider now a charged quantum particle restricted to move on a ring of radius r outside the solenoid. Its Hamiltonian is

$$H = \frac{1}{2m}(p_\phi + eA_\phi)^2 = \frac{1}{2mr^2}\left(-i\hbar\frac{\partial}{\partial\phi} + \frac{e\Phi}{2\pi}\right)^2. \tag{6}$$

Prove that the energy eigenstates are

$$\frac{1}{\sqrt{2\pi r}} e^{in\phi} \tag{7}$$

and that the eigenenergies are

$$E_n = \frac{\hbar^2}{2mr^2}\left(n + \frac{\Phi}{\Phi_0}\right)^2 \qquad n \in \mathbb{Z}, \tag{8}$$

with Φ_0 the quantum of flux.

(c) What is the behaviour of the energy spectrum as a function of Φ? Imagine that you prepare a particle in the ground state $n = 0$ for $\Phi = 0$. What happens to the quantum state of the particle if you adiabatically increase Φ until reaching $\Phi = \Phi_0$?